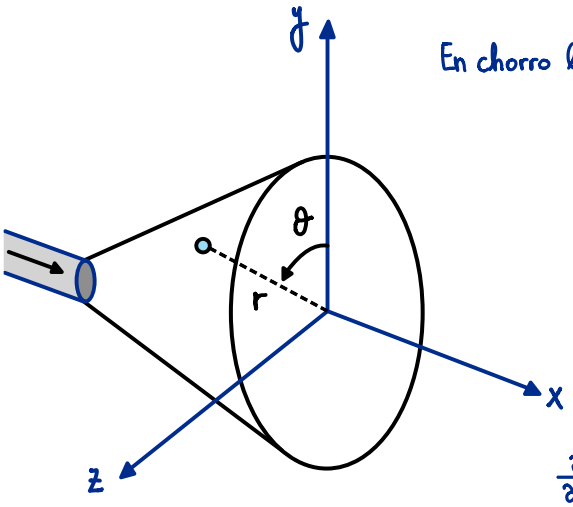
Chorro Axilsimétrico Laminar



En chorro lejano $\left(\frac{x}{R} >> 1\right)$: $\frac{\partial \blacksquare}{\partial \theta} = 0$

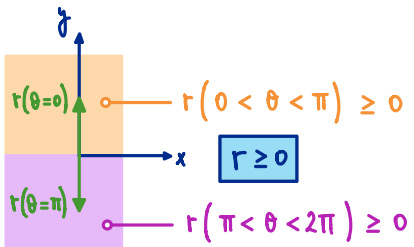
Continuidad : $\frac{\partial}{\partial x}(r \cdot u) + \frac{\partial}{\partial r}(r \cdot v) = 0$ (I)

$ECdM_x$: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu r \frac{\partial u}{\partial r}\right)$ (II)

$u \cdot (I) + r \cdot (II)$

$$\frac{\partial}{\partial x}(r \cdot u \cdot u) + \frac{\partial}{\partial r}(r \cdot u \cdot v) = \frac{\partial}{\partial r}\left(\nu r \frac{\partial u}{\partial r}\right)$$

FORMA CONSERVATIVA
(podemos integrar transversalmente)



Integramos transversalmente :

$$\frac{d}{dx}\int_0^\infty r u^2 dr + \underbrace{r \cdot u \cdot v\Big|_0^\infty}_{0} = \underbrace{\nu r \frac{\partial u}{\partial r}\Big|_0^\infty}_{0} \longrightarrow \boxed{\int_0^\infty u^2 r\, dr = I \ (\text{cte } \forall x)}$$

$r = 0 \quad v = 0 \qquad r = 0 \quad \frac{\partial u}{\partial r} = 0$

$r \to \infty \begin{cases} u \to 0 \\ v \to 0 \end{cases} \qquad r \to \infty \quad \frac{\partial u}{\partial r} \to 0$

Eliminamos $\nu \longrightarrow x, \frac{r}{\sqrt{\nu}}, u, \frac{v}{\sqrt{\nu}}$

Reescribiendo, el problema queda :

$$\frac{\partial}{\partial x}\left(\frac{r}{\sqrt{\nu}} u\right) + \frac{\partial}{\partial\left(\frac{r}{\sqrt{\nu}}\right)}\left(\frac{r}{\sqrt{\nu}}\frac{v}{\sqrt{\nu}}\right) = 0$$

$$u\frac{\partial u}{\partial x} + \frac{v}{\sqrt{\nu}}\frac{\partial u}{\partial\left(\frac{r}{\sqrt{\nu}}\right)} = \frac{1}{\frac{r}{\sqrt{\nu}}}\frac{\partial}{\partial\left(\frac{r}{\sqrt{\nu}}\right)}\left[\frac{r}{\sqrt{\nu}}\frac{\partial u}{\partial\left(\frac{r}{\sqrt{\nu}}\right)}\right]$$

$\frac{r}{\sqrt{\nu}} = 0 : \frac{v}{\sqrt{\nu}} = 0 \ ; \ \frac{\partial u}{\partial\left(\frac{r}{\sqrt{\nu}}\right)} = 0$

$\frac{r}{\sqrt{\nu}} \to \infty : \ u \to 0$

$$\int_0^\infty u^2 \frac{r}{\sqrt{\nu}}\, d\left(\frac{r}{\sqrt{\nu}}\right) = \frac{I}{\nu}$$

DIFERENCIA CON CHORRO PLANO

Introducimos la función de corriente Ψ:

$$r u = \frac{\partial \Psi}{\partial r} \longrightarrow u = \frac{1}{r}\frac{\partial \Psi}{\partial r} \longrightarrow u = \frac{1}{\frac{r}{\sqrt{J}}}\frac{\partial\left(\frac{\Psi}{J}\right)}{\partial\left(\frac{r}{\sqrt{J}}\right)}$$

SE ADIMENSIONALIZA CON J Y NO CON $\sqrt{J}$ COMO SE HACÍA EN CHORRO PLANO

$$r v = -\frac{\partial \Psi}{\partial x} \longrightarrow v = -\frac{1}{r}\frac{\partial \Psi}{\partial x} \longrightarrow \frac{v}{\sqrt{J}} = -\frac{1}{\frac{r}{\sqrt{J}}}\frac{\partial\left(\frac{\Psi}{J}\right)}{\partial x}$$

$$\longrightarrow \frac{\Psi}{J} = F\left(x, \frac{r}{\sqrt{J}}, \frac{I}{J}\right)$$

DUPLA DE ADIMENSIONALIZACIÓN

Ecuaciones de dimensiones:

$$[x] = [r] = L$$

$$\left[\frac{r}{\sqrt{J}}\right] = \frac{L}{\left(L^2 T^{-1}\right)^{1/2}} = T^{1/2}$$

$$\left[\frac{\Psi}{J}\right] = \frac{L^3 T^{-1}}{L^2 T^{-1}} = L$$

$$\left[\frac{I}{J}\right] = \frac{L^2 T^{-2} L^2}{L^2 T^{-1}} = L^2 T^{-1}$$

Adimensionalizamos $\frac{\Psi}{J}$ y $\frac{r}{J}$:

$$\frac{\frac{\Psi}{J}}{x^\alpha \left(\frac{I}{J}\right)^\beta} \longrightarrow \frac{L}{L^\alpha L^{2\beta} T^{-\beta}} \longrightarrow \begin{cases} 1 = \alpha + 2\beta \\ -\beta = 0 \end{cases} \longrightarrow \begin{cases} \alpha = 1 \\ \beta = 0 \end{cases} \longrightarrow \frac{\Psi}{Jx}$$

$$\frac{\frac{r}{\sqrt{J}}}{x^\alpha \left(\frac{I}{\sqrt{J}}\right)^\beta} \longrightarrow \frac{T^{1/2}}{L^\alpha L^{2\beta} T^{-\beta}} \longrightarrow \begin{cases} 0 = \alpha + 2\beta \\ \frac{1}{2} = -\beta \end{cases} \longrightarrow \begin{cases} \alpha = 1 \\ \beta = -\frac{1}{2} \end{cases} \longrightarrow \frac{r\sqrt{I}}{Jx}$$

Por tanto:

$$\frac{\Psi}{Jx} = f\left(\frac{r\sqrt{I}}{Jx}\right) = f(\eta) \quad , \text{ donde } \eta = \frac{r\sqrt{I}}{Jx} \text{ es la VARIABLE DE SEMEJANZA}$$

Hay que comprobar que la estructura de semejanza sea compatible con la ECdMx (pues la ecuación de continuidad se satisface automáticamente al introducir la función de corriente), lo que quiere decir que la ECdMx debe pasar de ser EDP(x, r) a ser EDO(η).

$$\psi = \nu x\, f(\eta) \;;\; \eta = \frac{r\sqrt{I}}{\nu x} \longrightarrow \frac{\partial \eta}{\partial x} = -\frac{\eta}{x} \;;\; \frac{\partial \eta}{\partial r} = \frac{\sqrt{I}}{\nu x}$$

Sustituimos en la ECdM$_x$: $u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu r \frac{\partial u}{\partial r}\right)$

u

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[\nu x f(\eta)\right] = \frac{1}{r}\nu x \underbrace{\frac{df}{d\eta}}_{f'}\frac{\partial \eta}{\partial r} = \frac{\nu x}{r} f' \frac{\sqrt{I}}{\nu x} = \frac{\sqrt{I}}{r} f' \longrightarrow \boxed{u = \frac{I}{\nu x}\frac{1}{\eta} f'}$$

v

$$v = -\frac{1}{r}\frac{\partial \psi}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial x}\left[\nu x f(\eta)\right] = -\frac{1}{r}\nu f - \frac{\nu x}{r} f' \frac{\partial \eta}{\partial x} = -\frac{\nu}{r} f + \frac{\nu x}{r} f' \frac{\eta}{x} = -\frac{\nu}{r}\left(f - \eta f'\right) =$$

$$= -\frac{\sqrt{I}}{\eta x}\left(\frac{f}{\eta} - f'\right) \longrightarrow \boxed{v = \frac{\sqrt{I}}{x}\left(f' - \frac{f}{\eta}\right)}$$

$\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{I}{\nu x}\frac{1}{\eta} f'\right) = -\cancel{\frac{I}{\nu x^2}\frac{1}{\eta} f'} + \cancel{\frac{I}{\nu x}\frac{1}{\eta^2}\frac{\eta}{x} f'} - \frac{I}{\nu x}\frac{1}{\eta} f'' \frac{\eta}{x} \longrightarrow \boxed{\frac{\partial u}{\partial x} = -\frac{I}{\nu x^2} f''}$$

$\frac{\partial u}{\partial r}$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(\frac{I}{\nu x}\frac{1}{\eta} f'\right) = -\frac{I}{\nu x}\frac{1}{\eta^2}\frac{\sqrt{I}}{\nu x} f' + \frac{I}{\nu x}\frac{1}{\eta} f'' \frac{\sqrt{I}}{\nu x} \longrightarrow \boxed{\frac{\partial u}{\partial r} = \frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left(\eta f'' - f'\right)}$$

$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} = \frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left(\eta f'' - f'\right) + \frac{\eta \nu x}{\sqrt{I}}\left\{-\frac{I\sqrt{I}}{(\nu x)^2}\frac{2}{\eta^3}\frac{\sqrt{I}}{\nu x}\left(\eta f'' - f'\right) + \right.$$

$$\left. + \frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left[\cancel{\frac{\sqrt{I}}{\nu x} f''} + \eta f''' \frac{\sqrt{I}}{\nu x} - \cancel{f'' \frac{\sqrt{I}}{\nu x}}\right]\right\} = \frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left(\eta f'' - f'\right) - \frac{2I\sqrt{I}}{\eta(\nu x)^2} f'' +$$

$$+ \frac{2I\sqrt{I}}{\eta^2(\nu x)^2} f' + \frac{I\sqrt{I}}{(\nu x)^2} f''' \longrightarrow \boxed{\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{I\sqrt{I}}{\eta^3(\nu x)^2}\left(\eta f' - \eta^2 f'' + \eta^3 f'''\right)}$$

Sustituyendo:

$$\frac{\nu}{r}=\frac{\sqrt{I}}{\eta x}$$

$$-\frac{I}{\nu x}\frac{1}{\eta}f'\frac{I}{\nu x^2}f''+\frac{\sqrt{I}}{x}\frac{1}{\eta}\left(f'-\frac{f}{\eta}\right)\frac{I\sqrt{I}}{(\nu x)^2}\frac{1}{\eta^2}\left(\eta f''-f'\right)=\frac{\sqrt{I}}{\nu x}\frac{I\sqrt{I}}{\eta^3(\nu x)^2}\left(\eta f'-\eta^2 f''+\eta^3 f'''\right)$$

$$-\cancel{\frac{I^2}{\eta\nu^2x^3}f'f''}+\cancel{\frac{I^2}{\eta\nu^2x^3}f'f''}-\frac{I^2}{\eta\nu^2x^3}(f')^2-\frac{I^2}{\eta\nu^2x^3}ff''+\frac{I^2}{\eta\nu^2x^3}ff'=\frac{I^2}{\eta\nu^2x^3}f'-\frac{I^2}{\eta\nu^2x^3}f''+\frac{I^2}{\eta\nu^2x^3}f'''$$

$$-\eta(f')^2-\eta f'f''+ff'=f'-\eta f''+\eta^2 f'''$$

$$\boxed{\frac{1}{\eta^2}\left[ff'-\eta(f')^2-\eta f'f''\right]=\left(f''-\frac{1}{\eta}f'\right)'}$$

Condiciones de contorno:

$$\eta=0:\ f=0\ ;\ f'=0$$

~~$\eta\to\infty:\ f'\to 0$~~

$$\int_0^\infty\frac{1}{\eta}(f')^2d\eta=1$$

Incluida en esta

Por tanto, el problema queda:

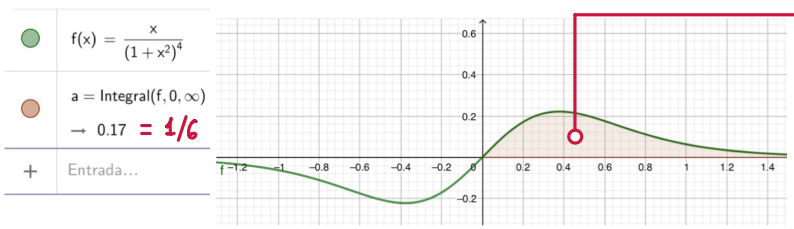
$$\boxed{\begin{aligned}&\frac{1}{\eta^2}\left[ff'-\eta(f')^2-\eta f'f''\right]=\left(f''-\frac{1}{\eta}f'\right)'\\&\eta=0:\ f=0\ ;\ f'=0\\&\int_0^\infty\frac{1}{\eta}(f')^2d\eta=1\end{aligned}}$$

La solución al problema es:

$$f=\frac{4(\alpha\eta)^2}{1+(\alpha\eta)^2}\qquad f'=\frac{8\alpha^2\eta\left[1+(\alpha\eta)^2\right]-4(\alpha\eta)^2 2\alpha^2\eta}{\left[1+(\alpha\eta)^2\right]^2}=\frac{8\alpha^2\eta}{\left[1+(\alpha\eta)^2\right]^2}$$

El valor de "α" se obtiene de aplicar la condición de contorno que falta:

$$\int_0^\infty\frac{1}{\eta}(f')^2d\eta=1\longrightarrow\int_0^\infty\frac{64\alpha^4\eta^2}{\left[1+(\alpha\eta)^2\right]^4}d\eta=1\longrightarrow 64\alpha^2\underbrace{\int_0^\infty\frac{\alpha\eta\,d(\alpha\eta)}{\left[1+(\alpha\eta)^2\right]^4}}_{\frac{1}{6}\ (\forall\alpha\in\mathbb{R})}=1$$



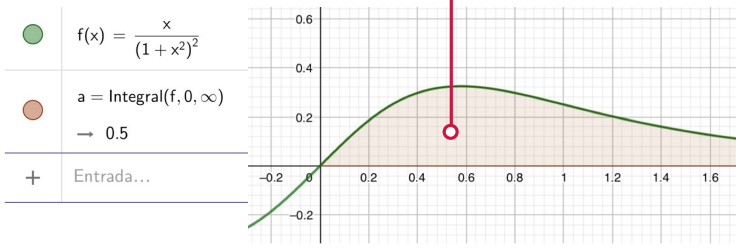
$$\alpha=\frac{1}{4}\sqrt{\frac{3}{2}}$$

Por tanto:

$$f = \frac{12\eta^2}{32 + 3\eta^2} \qquad f' = \frac{768\,\eta}{(32 + 3\eta^2)^2} \xrightarrow{u = \frac{I}{\sqrt{x}}\frac{1}{\eta}f'} u = \frac{768\,I}{\sqrt{x}}(32 + 3\eta^2)^{-2}$$

Velocidad en el centro del chorro: $u_{máx} = u(\eta = 0) = \frac{3I}{4\sqrt{x}}$

Flujo a través de una sección "x" del chorro:

$$Q = \int_{x=cte} \underbrace{\vec{v}\cdot\vec{n}}_{\vec{i}}\, dA = 2\pi\int_0^\infty u\,r\,dr = 2\pi\int_0^\infty \frac{768\,I}{\sqrt{x}}(32 + 3\eta^2)^{-2}\,\frac{\sqrt{x}\,\eta}{\sqrt{I}}\,d\left(\frac{\sqrt{x}\,\eta}{\sqrt{I}}\right) = 1536\,\pi\sqrt{x}\int_0^\infty \frac{\eta\,d\eta}{(32 + 3\eta^2)^2} =$$

$$= \frac{3}{2}\pi\sqrt{x}\int_0^\infty \frac{\eta\,d\eta}{\left(1 + \frac{3}{2}\eta^2\right)^2} = \frac{3}{2}\pi\sqrt{x}\,\frac{32}{3}\int_0^\infty \underbrace{\frac{\sqrt{\frac{3}{32}}\,\eta\,d\left(\sqrt{\frac{3}{32}}\,\eta\right)}{\left[1 + \left(\sqrt{\frac{3}{32}}\,\eta\right)^2\right]^2}}_{\frac{1}{2}} \longrightarrow \boxed{Q = 8\pi\sqrt{x}}$$